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FORMULATION OF THE PROBLEM OF THE FLUTTER OF A SHELL OF REVOLUTION AND A SHALLOW SHELL IN A HIGH-VELOCITY SUPERSONIC GAS FLOW†

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Aero-elastic vibration is investigated in the case of a shallow shell of revolution or a cylindrical panel, which respectively occupy a part of a thin cylindrical body or a thin profile, in a high-velocity supersonic gas flow at zero angle of attack. Particular attention is paid to finding the pressure interaction and this problem is solved within the framework of the law of plane sections in boundary-layer theory. An expression is obtained which refines and supplements the well-known formula of "piston" theory. A linearized formulation of the problem of the panel flutter of a shallow shell is presented. Using the example of a plate located on one of the sides of a wedge, it is shown that the formula of "piston" theory is complemented with a term which has the meaning of a compressive force in the plane of the plate. It is shown that, when account is taken of this term, there is a reduction in the critical flow velocity.

In the overwhelming majority of cases investigations of panel flutter are carried out using the formulae of "piston" theory for the pressure of the interaction between the vibrating panel and the gas flow. The drawbacks of such an approach in the general case have been noted earlier [1], and it has been proposed that the interaction pressure should be determined in the context of the law of plane sections [2, 3] from the solution of a perturbation problem (the source of these perturbations being the bending of the panel) when no account is taken of the reflection of the perturbations from the shock wave. Final relations were obtained in [1] after making quite considerable simplifications, and a further analysis of these equations is necessary.

1. DERIVATION OF THE BASIC RELATIONS

Consider a thin, axially symmetric body or profile in a high-velocity supersonic gas flow at a zero angle of attack. The velocity vector of the flow is directed along the body axis (orthogonal to the edge of the profile). The origin of the orthogonal system of coordinates coincides with the body vertex (with the edge of the profile), the x axis is directed along the velocity vector, the y axis is directed along the edge of the profile and the z axis is such that the system of coordinates is a right-hand system (in the case of a body, the direction of the y axis is arbitrary).

We will initially assume that the deformed part of the surface of the body or the profile occupies a domain $[x_1, x_2]$ and we shall be concerned with the axially symmetric flexures of a shell of revolution or the cylindrical bending of a shallow cylindrical shell. Suppose that, in the undeformed state, the equation of the generatrix is

$$z_1 = kx + \varphi(x)$$

where $|\varphi(x)/(kx)| \ll 1$ is a body of revolution which differs only slightly from a cone or a profile which differs only slightly from a wedge. In the deformed part, we shall then have $z = kx + \varphi(x) - \tilde{w}(x, t)$, where $\tilde{w} = w\cos(\mathbf{n}, z)$ and $\mathbf{n}$ is the outward normal to the surface z_1. To the same accuracy to which the law of plane sections holds, we may put $\tilde{w} \equiv w$ and, hence, finally

$$z = kx + \varphi(x) - w(x, t) \tag{1.1}$$

By the law of plane sections, the gas state in the domain between the body and the shock wave (SW) is determined from the solution of the problem of unsteady plane flow in the plane $x = \upsilon_x t$ which is caused by the expansion of a piston in accordance with the law

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$$z(t) = kv_x t + \varphi(v_x t) - w(v_x t, t) \tag{1.2}$$

We will construct the solution of this problem by expansion with respect to a small parameter, the ratio of the gas densities upstream and downstream of the SW [4]

$$\frac{\rho^0}{\rho^*} = \frac{\gamma - 1}{\gamma + 1}\left[1 + \frac{2a_0^2}{(\gamma - 1)D^2}\right] \equiv \varepsilon a(D)$$

where a_0 is the velocity of sound in the free stream and D is the propagation velocity of the SW. If ε is properly assumed to be the expansion parameter, the smallness requirement $(\rho^0/\rho^*)^2 \ll 1$ implies the condition $a(D) \sim 1$, which will henceforth be used.

We will now estimate the value of the flow velocity v_x which guarantee the above inequality. It follows from the conclusion $\rho/\rho^* \ll 1$ that $a_0^2/D^2 \ll 1$ and, as a consequence of the condition $|(\varphi - w)/(kx)| \ll 1$ for the velocity of the SW, the estimate $D \sim \delta k v_x$, $\delta \gtrsim 1$ holds and therefore, finally, we must have $(\delta kM)^2 \gg 1$.

We now introduce Lagrangian variables: the time t and the coordinate z so that $dz = \rho^0 r^{\mu-1} dr$, where ρ^0 is the initial density, $\mu = 2$ in the case of cylindrical waves and $\mu = 1$ in the case of plane waves. The required functions are the distance of the particles from the axis or the plane of symmetry $\varsigma = \varsigma(t, z)$, the pressure $p = p(t, z)$ and the density $\rho = \rho(t, z)$.

The equations of motion, conservation of mass and energy have the form

$$\frac{\partial^2 \varsigma}{\partial t^2} = -\varsigma^{\mu-1}\frac{\partial p}{\partial z}; \quad \frac{\partial \varsigma}{\partial z} = \frac{1}{\rho \varsigma^{\mu-1}}; \quad \frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{1.3}$$

Putting

$$\varsigma = \varsigma_0 + \varepsilon\varsigma_1 + \ldots, \quad p = p_0 + \varepsilon p_1 + \ldots, \quad \rho = \varepsilon^{-1}\rho_0 + \rho_1 + \ldots$$

and substituting into (1.3), we obtain a system for the function of the zeroth approximation which is easily integrated

$$\varsigma_0 = \varsigma_0(t), \quad p_0 = P(t) - z\varsigma_0^{1-\mu}\frac{\partial^2 \varsigma_0}{\partial t^2}, \quad \rho_0 = \frac{p_0^{1/\gamma}}{\vartheta_0(z)}$$

Here, $\varsigma_0(t)$, $P(t)$, $\vartheta_0(z)$ are as yet arbitrary functions.

For the functions of the first approximation, we have

$$\varsigma_1 = \frac{1}{\varsigma_0^{\mu-1}}\int_{z^*}^{z}\frac{\vartheta_0(z)}{p_0^{1/\gamma}}dz + \varsigma_1^*(t)$$

$$p_1 = (\mu - 1)\frac{\partial^2 \varsigma_0}{\partial t^2}\frac{1}{\varsigma_0^{\mu}}\int_{z^*}^{z}\varsigma_1 dz - \frac{1}{\varsigma_0^{\mu-1}}\int_{z^*}^{z}\frac{\partial^2 \varsigma_1}{\partial t^2}dz + p_1^*(t) \tag{1.4}$$

$$\frac{p_1}{p_0} - \gamma\frac{\rho_1}{\rho_0} = \vartheta_1(z)$$

Here ς_1^*, p_1^*, ϑ_1 are arbitrary functions and the quantity z^* will be determined later.

We assume that $\varsigma_0(t)$ determines the law of motion of the SW and, then, we shall have $z = z^* = \rho^0\varsigma_0^{\mu}(t)/\mu$ in it. On substituting the expansions adopted above into the conditions in the SW, we arrive at the following relations: when $z = z^* = \rho^0\varsigma_0^{\mu}/\mu$, we must have

$$\varsigma_0 = \varsigma_0(t), \quad p_0 = \frac{2}{\gamma + 1}\rho^0\dot{\varsigma}_0^2, \quad \rho_0 = \frac{\rho^0}{a(\dot{\varsigma}_0)} \tag{1.5}$$

$$\varsigma_1 = 0, \quad p_1 = -p^0, \quad \rho_1 = 0$$

Here p^0 is the pressure in the unperturbed flow and a dot over a function indicates a derivative with respect to time.

From (1.5), we now determine the arbitrary functions occurring in the solution and introduce the variable τ by the formula $z = \rho^0\varsigma_0^\mu(\tau)/\mu$. The functions required in the subsequent treatment take the form

$$p_0 = \frac{2\rho^0}{\gamma+1}\dot{\varsigma}_0^2 + \frac{1}{\mu}\rho^0\varsigma_0\ddot{\varsigma}_0 - \ddot{\varsigma}_0\varsigma_0^{1-\mu}z$$

$$\varsigma_1 = -\frac{1}{\varsigma_0^{\mu-1}}\int_\tau^t a(\dot{\varsigma}_0(\xi))\psi(t,\xi)\varsigma_0^{\mu-1}(\xi)\dot{\varsigma}_0^{1+2/\gamma}(\xi)d\xi \equiv \varsigma_1(t,\tau)$$

$$p_1 = -(\mu-1)\frac{\rho^0\ddot{\varsigma}_0}{\varsigma_0^\mu}\int_\tau^t \varsigma_1(t,\xi)\varsigma_0^{\mu-1}(\xi)\dot{\varsigma}_0(\xi)d\xi + \frac{\rho^0}{\varsigma_0^{\mu-1}}\int_\tau^t \frac{\partial^2\varsigma_1}{\partial t^2}\varsigma_0^{\mu-1}(\xi)\dot{\varsigma}_0(\xi)d\xi - p^0 \tag{1.6}$$

$$\psi(t,\tau) = \left[\dot{\varsigma}_0^2(t) + \frac{1}{\mu}\varsigma_0(t)\ddot{\varsigma}_0(t)\left(1 - \frac{\varsigma_0^\mu(\tau)}{\varsigma_0^\mu(t)}\right)\right]^{-1/\gamma}$$

Thus, the solution is expressed in terms of a law of motion of the SW $\varsigma_0(\tau)$ which is as yet unknown. We find the equation for determining it from the condition on the piston: when $\tau = 0$ ($z = 0$), the equality (if we confine ourselves in the solution to terms which are linear with respect to ε) $\varsigma(t) = \varsigma_0(t) + \varepsilon\varsigma_1(t, 0) = z(t)$ must be satisfied which, by using the notation $\zeta_1(t, 0) = -F\{t, \xi; \zeta_0(t), \zeta_0(\xi)\}$, we write in the form

$$\zeta_0(t) = \varepsilon F\{\ldots\} + z(t) \tag{1.7}$$

The function $z(t)$ is given by expression (1.2).

The functional F is extremely non-linear and an analytical solution of Eq. (1.7) is therefore practically impossible. However, the existence of a small parameter indicates the possibility of obtaining an approximate solution by the method of successive approximations

$$\varsigma_0^{(0)}(t) = z(t), \quad \varsigma_0^{(n+1)}(t) = \varepsilon F\{t,\xi;\varsigma_0^{(n)}(t),\varsigma_0^{(n)}(\xi)\} + z(t) \tag{1.8}$$

Without considering the general case, we shall merely point out one consideration in favour of the possible convergence of the sequence (1.8). In the case when $\varphi = w = 0$, Eq. (1.7) has an exact solution $\varsigma_0(t) = Dt$, in which D is found from the quadratic equation $D = \varepsilon Da(D)/\mu + u$, $u = k\upsilon_x$. The sequence (1.8) leads to an inertial process for determining D

$$D(0) = u, \quad D^{(n+1)} = u + \varepsilon D^{(n)}a(D^n)/\mu$$

which converges when $a_0/u < [(\gamma + 1)(\mu + \varepsilon)/2]^{1/2}$ [5] and, in particular, when $\mu = 1$, whence it follows that $k\gamma M > 1$ which is in agreement with the estimate obtained above.

In the linear theory of thin shallow shells it is assumed that

$$(\varphi/ut)^2 \ll 1, \quad (w/\varphi)^2 \ll 1$$

and it may therefore be expected that the sequence (1.8) will converge subject to a condition that differs only slightly in a certain sense from the condition presented above.

We shall make some estimates beforehand. Suppose l is the characteristic dimension of the shell in the flow direction. Then, $t_1 = l/\upsilon_x$ will be the characteristic time for the flow around the shell. The characteristic time of the vibration of the shell is $t_2 = l_2/(\mathrm{ch})$, where $c = (E/\tilde{\rho})^{1/2}$ E and $\tilde{\rho}$ are Young's modulus and the density of the shell material and h is its thickness; hence

$$t_1/t_2 = \mathrm{ch}/(v_x\, l^2 \lesssim 1$$

Next

$$\dot{\varphi}/u \sim \varphi/(t_1 u) \ll 1, \quad \varsigma_0\ddot{\varphi}/\dot{\varsigma}_0^2 \sim \varphi/(kl) \ll 1;$$

since $(w/\varphi)^2 \ll 1$, the estimates

$$\dot{w}/u \ll 1, \quad \varsigma_0\ddot{w}/\dot{\varsigma}_0^2 \ll 1$$

will hold all the more.

We will now calculate the first approximation in (1.8). We obtain

$$\varsigma_0^{(1)}(t)=\varepsilon F\{t,\xi;z(t),z(\xi)+z(t) \tag{1.9}$$

We expand all expressions of the type $1+(\varphi-w)/(ut)$ and similar expressions, which are contained in $\varsigma_1(t,\tau)$ and $F\{\ldots\}$ to differing powers, in series and carry out the necessary operations. We finally write equality (1.9) in a form which only contains linear terms with respect to $\varphi-w=W$ and the corresponding derivatives

$$\varsigma_0^{(1)}(t)=ut+\varepsilon uta(u)+\left(1+\frac{\varepsilon}{\mu}a(u)\right)W(t)-\frac{2\varepsilon}{\mu\gamma}a(u)\dot W(t)t-\frac{\varepsilon}{2\mu^2\gamma}a(u)\ddot W(t)t^2+$$

$$+\frac{2\varepsilon}{\gamma}\left(a(u)-\frac{2\gamma}{\gamma-1}\frac{a_0^2}{u^2}\right)t^{1-\mu}\int_0^t\tau^{\mu-1}\dot W(\tau)d\tau \tag{1.10}$$

It is necessary to assume that $\varphi(\tau)\equiv\varphi(v_x\tau)$, $w(\tau)\equiv w(v_x\tau,t)$ in all the formulae.

We will now calculate the second approximation in (1.8), retaining only linear terms with respect to ε of the type εW, εW and analogous terms. It can be shown that $\varsigma_0^{(2)}$ will differ from $\varsigma_0^{(1)}$ solely in the fact that the substitutions

$$u_1t=(u+\varepsilon ua(u)/\mu)t\to u_2t=(u+\varepsilon u_1a(u_1)/\mu)t,\qquad a(u)\to a(u_1)$$

have to be made in (1.10).

A similar structure of the solution is obviously preserved in any approximation. The iterative process for u_n, as was established above, converges to D and we therefore finally obtain

$$\varsigma_0(t)=Dt+(1+\varepsilon a(D)/\mu)W-\frac{2\varepsilon}{\mu\gamma}a(D)\dot Wt-\frac{\varepsilon}{2\mu^2\gamma}a(D)\ddot Wt^2+$$

$$+\frac{2\varepsilon}{\gamma}((1-\gamma)a(D)+\gamma)t^{1-\mu}\int_0^t\tau^{\mu-1}\dot W(\tau)d\tau \tag{1.11}$$

From (1.6), we determine $\varsigma_1(t,\tau)$

$$\varsigma_1(t,\tau)=-\frac{a(D)}{\mu}Dt\left(1-\frac{\tau^\mu}{t^\mu}\right)+\frac{2a(D)}{\mu\gamma}\dot Wt\left(1-\frac{\tau^\mu}{t^\mu}\right)-\frac{2}{\gamma}(1-\gamma)a(D)+\gamma)\int_\tau^t\tau^{\mu-1}\dot W(\tau)d\tau+$$

$$+(\mu-1)a(D)W\left(1-\frac{\tau^\mu}{t^\mu}\right)-a(D)\left(W(t)-W(\tau)\frac{\tau^{\mu-1}}{t^{\mu-1}}\right)+\frac{a(D)}{2\mu^2\gamma}\ddot Wt^2\left(1-2\frac{\tau^\mu}{t^\mu}+\frac{\tau^{2\mu}}{t^{2\mu}}\right) \tag{1.12}$$

The problem is, in fact, practically solved since p_1 can be calculated in terms of ς_0 and ς_1 using simple quadratures.

2. A SHALLOW SHELL AS PART OF THE SURFACE OF THE PROFILE

Substituting expressions (1.11) and (1.12) into (1.6), with $\mu=1$, we find the pressure on the piston

$$p\,|_{\tau=0}\cong(p_0+\varepsilon p_1)_{\tau=0}=\frac{2\rho^0D^2}{\gamma+1}-\varepsilon p^0+\frac{4\rho^0D\dot W}{\gamma+1}(1+2\varepsilon-\varepsilon a(D))+$$

$$+\rho^0D\ddot Wt\left[1-\varepsilon\frac{12a(D)}{\gamma(\gamma+1)}\right]+\ldots \tag{2.1}$$

The dots denote terms which are of a higher order of smallness compared with the preceding term and we therefore omit them.

On passing to the problem of the flow around a profile in an Eulerian system of coordinates associated with a fixed body, as was assumed above, we take account of the fact that

$$t = x/v_x, \quad \dot{W} = \partial W/\partial t + v_x \partial W/\partial x$$

Substituting this into (2.3), we obtain an expression for the pressure drop on the side surface of the profile

$$\Delta p = p - p^0 = \frac{2}{\gamma+1}(\rho^0 D^2 - \gamma p^0) - \frac{4\rho^0 D}{\gamma+1}(1+2\varepsilon-\varepsilon a(D))\left(\frac{\partial w}{\partial t} + v_x\frac{\partial w}{\partial x}\right) -$$
$$-\frac{\rho^0 Dx}{v_x}\left(1-\varepsilon\frac{12a(D)}{\gamma(\gamma+1)}\right)\left(\frac{\partial^2 w}{\partial t^2} + 2v_x\frac{\partial^2 w}{\partial t\partial x} + v_x^2\frac{\partial^2 w}{\partial x^2}\right) +$$
$$+\frac{4\rho^0 D v_x}{\gamma+1}(1+2\varepsilon-\varepsilon a(D))\frac{\partial\varphi}{\partial x} + \rho^0 D v_x x\frac{\partial^2\varphi}{\partial x^2} \tag{2.2}$$

We shall drop the assumption regarding cylindrical bending and assume that expression (2.2) holds in the general case when $\varphi = \varphi(x, y)$, $w = w(x, y, t)$. In order to describe the motions of the shell we shall use the simplest version of the linear theory [6]

$$D_0\Delta^2 w = \left(k_x + \frac{\partial^2 w}{\partial x^2}\right)h\frac{\partial^2\Phi}{\partial y^2} + \left(k_y + \frac{\partial^2 w}{\partial y^2}\right)h\frac{\partial^2\Phi}{\partial x^2} - 2h\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2\Phi}{\partial x\partial y} + \Delta p - \tilde{\rho}h\frac{\partial^2 w}{\partial t^2},$$
$$D_0 = \frac{Eh^3}{12(1-\nu^2)}, \quad \Delta^2\Phi + E\left(k_x\frac{\partial^2 w}{\partial y^2} + k_y\frac{\partial^2 w}{\partial x_2}\right) = 0 \tag{2.3}$$

Here, k_x, k_y are the principal curvatures, ν is Poisson's ratio of the shell material and Φ is the stress function. It is necessary to supplement system (2.3) with the appropriate boundary conditions.

In (2.3), we now isolate the ground (static) state $w_0(x, y)$, $\Phi_0(x, y)$ and put $w = w_0 + w^*$, $\Phi = \Phi_0 + \Phi^*$, where "small" perturbations of the ground state are denoted by an asterisk. Substituting into (2.3) we obtain a linearized system in small perturbations

$$D_0\Delta^2 w^* = h\left(k_x\frac{\partial^2\Phi^*}{\partial y^2} + k_y\frac{\partial^2\Phi^*}{\partial x^2}\right) + L(w^*, \Phi_0) + L(w_0, \Phi^*) -$$
$$-\frac{4\rho^0 D}{\gamma+1}(1+2\varepsilon-\varepsilon a(D))\left(\frac{\partial w^*}{\partial t} + v_x\frac{\partial w^*}{\partial x}\right) - \frac{\rho^0 Dx}{v_x}\left(1-\varepsilon\frac{12a(D)}{\gamma(\gamma+1)}\right)\times$$
$$\times\left(\frac{\partial^2 w^*}{\partial t^2} + 2v_x\frac{\partial^2 w^*}{\partial t\partial x} + v_x^2\frac{\partial^2 w^*}{\partial x^2}\right) - \tilde{\rho}h\frac{\partial^2 w^*}{\partial t^2}$$
$$\Delta^2\Phi^* + E\left(k_x\frac{\partial^2 w^*}{\partial y^2} + k_y\frac{\partial^2 w^*}{\partial x^2}\right) = 0$$
$$L(u,v) = \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial^2 v}{\partial x\partial y}$$

The term $L(w_0, \Phi^*)$ will most likely turn out to have a second-order effect on the solution w^*, Φ^* and it can be neglected: a final conclusion can be drawn after carrying out actual calculations.

We will now assume that the flow velocity vector makes an angle θ with the positive direction of the x axis so that

$$V = v\,\mathbf{n}^0 = \{v_x, v_y\}, \quad \mathbf{n}^0 = \{\cos\theta, \sin\theta\}$$

Expression (2.2) then becomes

$$\Delta p = \frac{2}{\gamma+1}(\rho^0 D - \gamma p^0) - \frac{4\rho^0 D}{\gamma+1}(1+2\varepsilon-\varepsilon a(D))\left(\frac{\partial w}{\partial t} + v\,\mathbf{n}^0\cdot\operatorname{grad} w\right) -$$

$$-\left\{\frac{\rho^0 Dx}{\upsilon_x}\left[1-\varepsilon\frac{12a(D)}{\gamma(\gamma+1)}\right]\right\}\left(\frac{\partial^2 w}{\partial t^2}+2\upsilon\frac{\partial}{\partial t}\mathbf{n}^0\cdot\operatorname{grad}w+\upsilon_x^2\frac{\partial^2 w}{\partial x^2}+2\upsilon_x\upsilon_y\frac{\partial^2 w}{\partial x\partial y}+\upsilon_y^2\frac{\partial^2 w}{\partial y^2}\right)+$$

$$+\frac{4\rho^0 D\upsilon}{\gamma+1}(1+2\varepsilon-\varepsilon a(D))\mathbf{n}^0\cdot\operatorname{grad}\varphi+\frac{\rho^0 Dx}{\upsilon_x}\left(1-\varepsilon\frac{12a(D)}{\gamma(\gamma+1)}\right)\times$$

$$\times\left(\upsilon_x^2\frac{\partial^2\varphi}{\partial x^2}+2\upsilon_x\upsilon_y\frac{\partial^2\varphi}{\partial x\partial y}+\upsilon_y^2\frac{\partial^2\varphi}{\partial y^2}\right)$$

In conclusion, we will give some estimates of the terms on the right-hand side of (2.2). The "combined mass" $(\rho^0 D_x/\upsilon_x)(1-12\varepsilon a(D))/(\gamma(\gamma+1))$ is comparable with the mass per unit length of the shell. Since the second factor is of the order of unity and $x \sim l$, we have

$$\tilde{\rho}h\upsilon_x/(\rho^0 Dl) \sim \rho h\upsilon_x/(\rho^0\upsilon_x \operatorname{tg}\alpha l)$$

and, subject to the usual limits on the change in the parameters, this ratio will be of the order of 10 to 10^3, that is, the term with the "combined mass" can be neglected in the first approximation. The characteristic time of the vibration of the shell is $t_2 = l^2/(\mathrm{ch})$ and the ratio

$$(2\upsilon_x\partial^2 w/\partial t\partial x)/(\upsilon_x^2\partial^2 w/\partial x^2) \sim \mathrm{ch}/(\upsilon_x l)$$

will therefore be of the order of 10^{-1} to 10^{-2}. Consequently, the main contribution to the "dynamic" load on the shell is made by the terms

$$\Delta P_{\mathrm{din}} = -\frac{4\rho^0 D}{\gamma+1}(1+2\varepsilon-\varepsilon a(D))\left(\frac{\partial w}{\partial t}+\upsilon_x\frac{\partial w}{\partial x}\right)-\rho^0 D\upsilon_x x\left(1-\varepsilon\frac{12a(D)}{\gamma(\gamma+1)}\right)\frac{\partial^2 w}{\partial x^2} \tag{2.4}$$

The first of these is the traditional term of "piston" theory, but with a coefficient which depends on the flow velocity in a rather complex way, while the second term has the meaning of a compressive normal force in the middle surface of the shell and, obviously, may have a substantial effect on the nature of the vibrations in the critical flutter velocity.

3. THE FLUTTER OF A PLATE

We will now consider the problem of a the flutter of a plate which occupies a domain G with a piecewise-smooth contour Γ in the (x, y) plane. Since the problem is linear, the vibration of the plate will be described by the equation

$$D_0\Delta^2 w = \Delta P_{\mathrm{din}} - \tilde{\rho}h\frac{\partial^2 w}{\partial t^2} \tag{3.1}$$

and the boundary conditions

$$x, y \in \Gamma : w = 0, \quad M(w) = 0 \tag{3.2}$$

where M is an operator which is well known in the theory of plates.

Suppose l is the characteristic size of the domain G and $t_0 = l^2\sqrt{\tilde{\rho}h/D_0}$. We will now introduce the dimensionless coordinates $x/l, y/l$ and time t/t_0, while retaining the previous notation for them, and then make the substitution $w = \psi(x, y)e^{\omega t}$. Then, taking (3.2) and expression (2.4) into account, we reduce problem (3.1) to the eigenvalue problem

$$\Delta^2\psi + A_2 M^2 x\frac{\partial^2\psi}{\partial x^2} + A_1 M^2\frac{\partial\psi}{\partial x} = \lambda\psi, \quad \lambda = -\omega^2 - A_0 M\omega \tag{3.3}$$

$$xy \in \Gamma, \quad \psi = 0, \quad M(\psi) = 0 \tag{3.4}$$

Here

$$A_0 = \frac{8\gamma\sqrt{3(1-\nu^2)}}{\gamma+1}\frac{p^0 c l^2}{E a_0 h^2}\operatorname{tg}\beta(1+2\varepsilon-\varepsilon a^*(\operatorname{tg}\beta))$$

$$A_1 = \frac{48\gamma(1-\nu^2)}{\gamma+1}\frac{p^0 l^3}{E h^3}\operatorname{tg}\beta(1+2\varepsilon-\varepsilon a^*(\operatorname{tg}\beta))$$

$$A_2 = 12\gamma(1-\nu^2)\frac{p^0 l^3}{E h^3}\operatorname{tg}\beta\left(1-\varepsilon\frac{12a^*(\operatorname{tg}\beta)}{\gamma(\gamma+1)}\right)$$

$$a^*(\operatorname{tg}\beta) = 1 + 2/[(\gamma-1)M^2\operatorname{tg}^2\beta]$$

The "slope of the shock wave" tg β is determined from the equation

$$\operatorname{tg}\beta = \operatorname{tg}\alpha + \varepsilon a^*(\operatorname{tg}\beta)\operatorname{tg}\beta$$

In the complex plane, λ, the domain of stable vibration lies inside the parabola

$$A_0^2 M^2 \operatorname{Re}\lambda = (\operatorname{Im}\lambda)^2$$

Consequently, the problem involves determining the eigenvalue which first falls within the parabola of stability. The velocity corresponding to this will be the critical flutter velocity.

The solution of Eq. (3.3) as an equation with variable coefficients is possible using approximate and numerical methods. In some of the numerical methods we note that there is a numerical–analytic algorithm without saturation [7], which has been specially developed for solving problems of this kind.

Remark. In the traditional formulation when $A_2 = 0$, the necessary condition for the motions Re $\lambda > 0$ to be stable is always satisfied [8]. In the case under consideration, since the operator x is negative-definite, this condition must be guaranteed $x\partial^2\psi/\partial x^2$ by means of additional constraints which are imposed on the parameters of the problem.

In the simple example of a rectangular hinge-supported plate, we obtain some results of a qualitative nature. Suppose that one of the sides of the plate is parallel to the edge of a wedge and is located at a distance x_0 (dimensionless) from it, and that the domain $G\{0 \leqslant x \leqslant 1/\beta_0,\ 0 \leqslant y \leqslant 1\}$. In the chosen system of coordinates, problem (3.3), (3.4) is written in the form

$$\begin{gathered}\Delta^2\psi + A_2 M^2(x_0+x)\frac{\partial^2\psi}{\partial x^2} + A_1 M^2\frac{\partial\psi}{\partial x} = \lambda\psi \\ x=0,\ x=\frac{1}{\beta_0}:\ \psi=0,\ \frac{\partial^2\psi}{\partial x^2}=0 \\ y=0,\ y=1:\ \psi=0,\ \frac{\partial^2\psi}{\partial y^2}=0\end{gathered} \tag{3.5}$$

It is well known [7, 9] that, when $\beta_0 \sim 1$, the two-term approximation of the Bubnov–Galerkin method in this simplest formulation of the problem when $A_2 = 0$ gives a result for the critical flutter velocity which is of acceptable accuracy. We shall make use of it and put

$$C = (C_1\sin\beta_0\pi x + C_2\sin 2\beta_0\pi x)\sin\pi y$$

After the usual procedure based on (3.5), we obtain a homogeneous system of linear algebraic equations in C_1, C_2 and we represent the roots of the characteristic equation of this system in the form

$$\lambda_{1,2} = \lambda' \pm i\lambda'' \tag{3.6}$$

$$\lambda' = \pi^4\left(1+5\beta_0^2+\frac{17}{2}\beta_0^4\right) - \frac{5}{2}\pi^2\beta_0^2 A_2 M^2\left(x_0+\frac{1}{2\beta_0}\right)$$

$$\lambda'' = \left(-\left[3\pi^4\beta_0^2(2+5\beta_0^2) - 3\pi^2\beta_0^2 A_2 M^2\left(x_0+\frac{1}{2\beta_0}\right)\right]^2 + \left(\frac{16\beta_0}{3}\right)^2 B_1 B_2 A_1^2 M^4\right)^{1/2}$$

$$B_1 = 1 - 8A_2/(3A_1),\quad B_2 = 1 + 2A_2/(3A_1)$$

From this, we obtain an equation for determining the critical value of the number M

$$A_0 M^2 \lambda' - \lambda^{2''} = 0 \tag{3.7}$$

We now determine M_0 from the condition that $\lambda'' = 0$, which corresponds to the critical flutter velocity, taking aerodynamic damping ($A_0 = 0$) into account; from (3.6) we have

$$\left[\sqrt{B_1 B_2} + \frac{9}{16}\beta_0 \pi^2 \left(x_0 + \frac{1}{2\beta_0} \right) \frac{A_2}{A_1} \right] A_1 M_0^2 = \frac{9}{16}\pi^4 \beta_0 (2 + 5\beta_0^2) \tag{3.8}$$

These simple estimates show that, when $a^* \sim 1$, the expression in the square brackets is greater than unity. In the case of "piston" theory $A_2 = 0$, $B_1 = B_2 = 1$ and it follows from (3.8) that

$$M_0^* = \tfrac{3}{4}\pi^2 [\beta(2 + 5\beta_0^2)/A_1]^{1/2} > M_0$$

Hence, "piston theory" gives an estimate of the critical flutter velocity that is too high using the approximate criterion $\operatorname{Im} \lambda = 0$. A qualitative analysis (when $\beta_0 = 1$ and $x_0 = 0$) shows that the exact criterion (3.7) also leads to the inequality $M_{cr} < M_{cr}^*$, where M_{cr}^* corresponds to the case when $A_2 = 0$.

The role of the last term in expression (2.5) for $\Delta P_{\rm din}$ increases in the case of plates elongated in the flow direction. In this case, it is necessary to make use of one of the versions of the geometrically non-linear theory of plates such as, for example, the Karman equations [6]. Suppose w_0 is a dimensionless flexure (with respect to $D_0/(Eh^2)$) and Φ_0 is a dimensionless stress function (with respect to D_0/h), which are obtained from the Karman system under "static" loading

$$\Delta P_{\rm st} = \frac{2}{\gamma+1}(\rho^0 D - \gamma p^0) - \frac{4\rho^0 D v_x}{\gamma+1}(1 + 2\varepsilon - \varepsilon a(D))\frac{\partial w_0}{\partial x} - \rho^0 D v_x x \left(1 - \varepsilon \frac{12a(D)}{\gamma(\gamma+1)}\right)\frac{\partial^2 w_0}{\partial x^2}$$

and we put $w = w_0 + w^*$, $\varphi = \varphi_0 + \varphi^*$.

In the system for "small" perturbations w^*, Φ^*, we discard the equation

$$\Delta^2 \Phi^* = -L(w_0, w^*)/2$$

and, in the remaining equation, we take no account of the effect of Φ^* and w^*. Then, when expression (2.3) is taken into account, we obtain the maximum simplified linearized formulation of the problem and, substituting $w^* = \psi(x, y) e\omega t$ into it, we arrive at the eigenvalue problem

$$\Delta^2 \psi + A_2 M^2 x \frac{\partial^2 \psi}{\partial x^2} + A_1 M^2 \frac{\partial \psi}{\partial x} - L(\Phi_0, \psi) = \lambda \psi$$

$$x, y \in \Gamma : \psi = 0, \quad M(\psi) = 0$$

which can be solved using a numerical-analytic algorithm without saturation.

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